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RELATIVE SPACECRAFT MOTION: A HAMILTONIAN APPROACH TO ECCENTRICITY PERTURBATIONS[*]

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This paper uses a Hamiltonian approach to find the effects of eccentricity perturbations on the linearized relative motion of spacecrafts described by Hill's equations. Perturbations to the constant canonical elements, obtained by a Hamiltonian treatment of the linearized relative motion, are considered. To begin with, the relative motion is described in an eccentric reference frame. Subsequently, the perturbing Hamiltonian is found in terms of the eccentricity. Next, a perturbation analysis is carried out via a variation of parameters procedure, generating a closed-form solution for variations about the eccentric reference orbit. Finally, using the orbit-averaged equations, the eccentricity effects on boundedness are discussed.

INTRODUCTION

Current mission plans for spacecraft formation flying have highlighted the need for improved dynamical formulations of the relative motion of spacecraft. The Clohessy-Wiltshire (C-W) equations were the first relative motion equations for the rendezvous of spacecrafts[1]. Linearizing the relative motion around a circular reference frame, the C-W equations express the relative motion in terms of Cartesian initial conditions. A major drawback of the C-W formulation is the difficulty solving for the motion under arbitrary perturbations. To overcome this problem, a substantial number of studies have examined the concepts for modeling motion under several perturbations[2,3].

One alternative approach utilizes inertial orbital elements to define the motion in terms of six constants of motion[4]. In Kasdin & Gurfil[5], closed form solutions for the relative motion are obtained to arbitrary order in the orbital elements. Nevertheless, this is still an inertial description of the motion.

In an alternative approach, Kasdin & Gurfil used a Hamiltonian formalism to derive a closed form solution for the motion relative to a circular orbit in terms of six canonical constants of the motion. Variational equations for these constants under conservative perturbations are then obtained via Hamilton's equations on the perturbation Hamiltonian. One appealing feature of this approach is that the motion is

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described entirely in terms of relative variables. No inertial measurements or references are necessary.

This paper uses this same formalism to examine the effects of the eccentricity of the reference orbit on the relative motion. First, the basic derivation of the canonical equations for relative motion about a circular reference orbit are summarized, including some recent modifications to make the canonical variables more physically sound. Next, the same procedure is used to find the Hamiltonian for motion relative to an eccentric reference orbit. Then, using a perturbation analysis for the Hamiltonian, the variational equations for the canonical elements under small eccentricity are obtained. Lastly, the effect of eccentricity on boundedness and periodicity for different initial conditions are discussed.

CANONICAL ANALYSIS OF RELATIVE MOTION IN CIRCULAR REFERENCE FRAME

Kasdin & Gurfil[5] used a circular rotating Euler-Hill reference frame, $\Re$, shown in Figure 1, with mean motion $n = \sqrt{\mu / a^3}$, where μ is the gravitational constant and a is the radius of the circular reference frame, to illustrate the Hamiltonian approach.

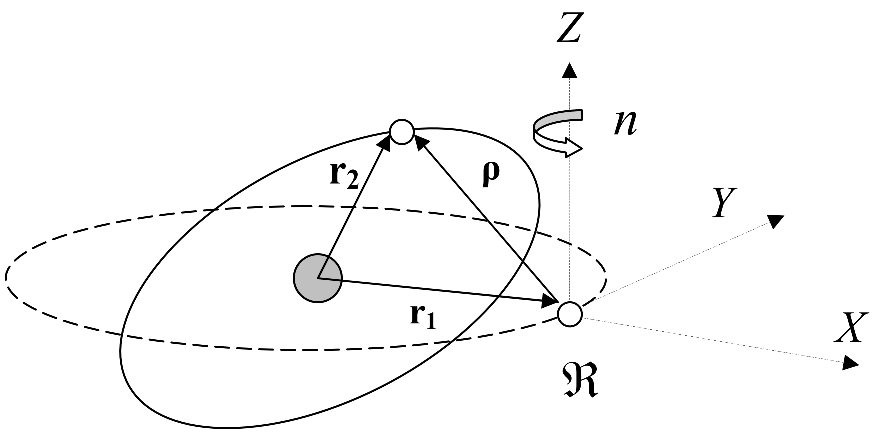

Figure 1 Relative motion in rotating Euler-Hill reference frame

A complete Lagrangian, using the gravitational potential energy, can be formulated for motion in relative motion frame $\Re$. The linearized equations of motion are found by first forming a low order Lagrangian via a 2nd order expansion of the potential in relative position, ρ. This Lagrangian is normalized by the reference orbit rate, n, and the reference orbit radius, a, and then used in a Legendre transformation to find the low-order Hamiltonian:

$$H^{(0)} = \frac{1}{2}(p_x + y)^2 + \frac{1}{2}(p_y - x - 1)^2 + \frac{1}{2}p_z^2 - \frac{3}{2} - \frac{3}{2}x^2 + \frac{1}{2}z^2 \tag{1}$$

where the canonical momenta are:

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - y$$
$$p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + x + 1 \tag{2}$$
$$p_z = \frac{\partial L}{\partial \dot{z}} = \dot{z}$$

By solving the Hamilton-Jacobi equation, we find new canonical constants of the motion, termed “epicyclic” elements:

$$\alpha_1 = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{3}{2}x^2$$
$$\alpha_2 = \frac{1}{2}\dot{z}^2 + \frac{1}{2}z^2$$
$$\alpha_3 = \dot{y} + 2x$$
$$Q_1 = -\tan^{-1}\left(\frac{3x + 2\dot{y}}{|\dot{x}|}\right) \tag{3}$$
$$Q_2 = \tan^{-1}\left(\frac{z}{|\dot{z}|}\right)$$
$$Q_3 = -(3\dot{y} + 6x)\tan^{-1}\left(\frac{3x + 2\dot{y}}{|\dot{x}|}\right) - 2|\dot{x}| + y$$

Defining a new action variable, $\alpha_1' = 2\alpha_1 + \frac{3\alpha_3^2}{2}$, the new low order Hamiltonian becomes:

$$H^{(0)} = \alpha_1 + \alpha_2 = \frac{\alpha_1'}{2} - \frac{3\alpha_3^2}{2} + \alpha_2 \tag{4}$$

Modifying the generating function accordingly, the equations for the new action-angle variables are obtained[5]:

$$\alpha_1' = \dot{x}^2 + 4\dot{y}^2 + 12x\dot{y} + 9x^2$$
$$\alpha_2' = \frac{1}{2}\dot{z}^2 + \frac{1}{2}z^2$$
$$\alpha_3' = \dot{y} + 2x$$
$$Q_1' = \frac{t}{2} + \beta_1' = -\frac{1}{2}\tan^{-1}\left(\frac{3x + 2\dot{y}}{|\dot{x}|}\right) \tag{5}$$
$$Q_2' = t + \beta_2' = \tan^{-1}\left(\frac{z}{|\dot{z}|}\right)$$
$$Q_3' = -\alpha_3' t + \beta_3' = -2|\dot{x}| + y$$

Dropping the primes for convenience, the relative motion in Cartesian coordinates can be expressed in terms of the new canonical elements in the following way:

$$
\begin{aligned}
x(t) &= 2\alpha_3 + \sqrt{\alpha_1}\sin(2Q_1) \\
y(t) &= Q_3 + 2\sqrt{\alpha_1}\cos(2Q_1) \\
z(t) &= 2\sqrt{\alpha_2}\sin(Q_2)
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
p_x(t) &= \sqrt{\alpha_1 - (-2\alpha_3 + x)^2} - y \\
p_y(t) &= 1 + \alpha_3 - x \\
p_z(t) &= \sqrt{2\alpha_2 - z^2}
\end{aligned} \tag{7}
$$

In this form, it is apparent that α_3 and Q_3 are responsible for in-plane secular drift. In their absence, the in-plane motion is periodic, with a 2:1 ellipse.

RELATIVE MOTION IN ECCENTRIC REFERENCE FRAME

To allow for a slightly eccentric reference orbit, we need a reference frame, $\Re'$, rotating with the angular velocity of a Keplerian orbit, $\dot{\theta}$, which is the time derivative of true anomaly(see Figure 2).

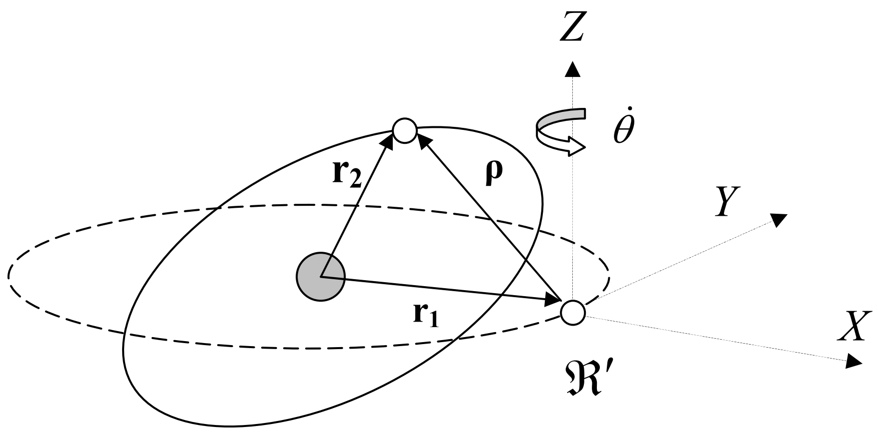

Figure 2 Relative motion in rotating $\Re'$ reference frame

Employing the coordinate system $\Re'$ and using a new variable, $r_c = \|\vec{r}_1\|$, the velocity of the follower can be written as:

$$
\vec{v} = \begin{bmatrix} \dot{x} + \dot{r}_c - \dot{\theta}y \\ \dot{y} + \dot{\theta}(x + r_c) \\ \dot{z} \end{bmatrix} \tag{8}
$$

The kinetic energy per unit mass can be found as:

$$K=\frac{1}{2}\|\bar{v}\|^2=\frac{1}{2}\{(\dot{x}+\dot{r}_c-\dot{\theta}\,y)^2+(\dot{y}+\dot{\theta}(x+r_c))^2+\dot{z}^2\} \tag{9}$$

Using the gravitational potential energy up to 2nd order in ρ, the potential energy is:

$$U=-\frac{\mu}{r_c}+\frac{\mu x}{r_c^{\,2}}-\frac{\mu x^2}{r_c^{\,3}}+\frac{\mu y^2}{2r_c^{\,3}}+\frac{\mu z^2}{2r_c^{\,3}} \tag{10}$$

Normalizing rates by the mean orbit rate, *n,* and relative distances by the semi-major axis, *a,* the low-order Lagrangian is obtained as follows:

$$\overline{L}=\frac{1}{2}\{(\dot{x}+\dot{r}_c-\dot{\theta}\,y)^2+(\dot{y}+\dot{\theta}(x+r_c))^2+\dot{z}^2\}+\frac{1}{r_c}-\frac{x}{r_c^{\,2}}+\frac{1}{2r_c^{\,3}}(2x^2-y^2-z^2) \tag{11}$$

The Legendre transformation, $H=\sum_i \dot{q}_i p_i - L$, gives us the complete Hamiltonian:

$$H=-\frac{\mu}{r_c}+\frac{\mu x}{r_c^{\,2}}+\frac{\mu}{2r_c^{\,3}}(-2x^2+y^2+z^2)$$
$$+\frac{1}{2}(p_{x'}^{\,2}+p_{y'}^{\,2}+p_{z'}^{\,2})+p_{x'}(-\dot{r}_c+y\dot{\theta})-p_{y'}(r_c+x)\dot{\theta} \tag{12}$$

θ and r_c are not independent variables, but rather predetermined functions of time and initial conditions. Thus, $p_{x'}$, $p_{y'}$, $p_{z'}$ are the only canonical momenta defined as:

$$p_{x'}=\frac{\partial L}{\partial \dot{x}}=\dot{x}+\dot{r}_c-y\dot{\theta}$$
$$p_{y'}=\frac{\partial L}{\partial \dot{y}}=\dot{y}+x\dot{\theta}+r_c\dot{\theta} \tag{12}$$
$$p_{z'}=\frac{\partial L}{\partial \dot{z}}=\dot{z}$$

ANALYSIS OF THE ECCENTRICITY PERTURBATIONS

The Hamilton-Jacobi equation for this system is unsolvable. However, when the eccentricity is small, the eccentric Hamiltonian approaches the circular Hamiltonian. As a result, we can conceive of the eccentric motion as a perturbation of the motion relative to a circular orbit of the same period. In this section, the eccentricity perturbation to the circular Hamiltonian is considered, and the Canonical Perturbation Theory is used to find the variations of the action-angle variables. The perturbing Hamiltonian, $H^{(1)}$, is by definition:

$$H^{(1)}=H-H^{(0)}=1+p_y-\frac{1}{r_c}-x+p_y x+\frac{x}{r_c^{\,2}}+x^2-\frac{x^2}{r_c^{\,3}}-p_x y$$
$$-\frac{y^2}{2}+\frac{y^2}{2r_c}-\frac{z^2}{2}+\frac{z^2}{2r_c}-p_x\dot{r}_c-\left(p_y(r_c+x)-p_x y\right)\dot{\theta} \tag{13}$$

Hamilton’s equations on the perturbing Hamiltonian give the evolution of the action-angle variables in time[6]:

$$
\begin{aligned}
\dot{\alpha}_i &= -\frac{\partial H^{(1)}}{\partial Q_i} \\
\dot{\beta}_i &= \frac{\partial H^{(1)}}{\partial \alpha_i} \\
\dot{Q}_i &= \frac{\partial H^{(0)}}{\partial \alpha_i} + \dot{\beta}_i
\end{aligned}
\tag{14}
$$

By substituting for x, y, z, p_x, p_y and p_z from Eqs. (6) and (7) and then applying Hamilton's equations, it is possible to find the variation equations. Our aim is to find the evolution of the system across time, and as a result, the time dependence of r_c and θ should be investigated before we proceed:

$$r_c = 1 - e\cos E \tag{15}$$

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right) \tag{16}$$

where E, the eccentric anomaly, is obtained from the Kepler's Equation:

$$M = n(t-\tau) = E - e\sin E \tag{17}$$

Here τ, the time of passage through the pericenter, is assumed to be zero, since it does not alter the problem's characteristics.

Kepler's Equation cannot be solved explicitly for time. We thus adopt the Fourier-Bessel series from Battin[7], and express r_c and θ in terms of eccentricity and time as follows:

$$\theta = M + 2\sum_{k=1}^{\infty}\frac{1}{k}\left[\sum_{n=-\infty}^{\infty} Jn(-ke)\left(\frac{1-\sqrt{1-e^2}}{e}\right)^{|k+n|}\right]\sin(kM) \tag{18}$$

$$r_c = 1 + \frac{e^2}{2} - 2e\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{dJ_k(ke)}{de}\sin(kM) \tag{19}$$

ORBIT-AVERAGED PERTURBATION ANALYSIS

To find an expression for the variation of parameters caused by the eccentric reference orbit, we substitute for r_c and θ from Eqs. (18) and (19) in the expression for $H^{(1)}$ in Eq. (13), and we expand up to second order in e:

$$
\begin{aligned}
H^{(1)} = {} & \frac{e}{2}\left[\left(-2 + 4x - 6x^2 - 2p_y(1+2x) + 4p_x y + 3y^2 + 3z^2\right)\cos(t) - 2p_x\sin(t)\right] + \\
& \frac{e^2}{4}\begin{bmatrix} 2p_y + 2x - 6x^2 + 3y^2 + 3z^2 + \\ \left(-4 + 10x - 2p_y(2+5x) + 10p_x y - 18x^2 + 9y^2 + 9z^2\right)\cos(2t) \\ -4p_x\sin(2t) \end{bmatrix} + O\left(e^3\right)
\end{aligned}
\tag{20}
$$

An expression for $H^{(1)}$ in terms of action-angle variables can be obtained by substituting for the Cartesian coordinates and momenta from Eqs. (6) and (7) into Eq. (20).

Because the long-term and secular behavior of the orbit is of great importance for many applications, this paper analyzes the orbit-averaged equations. In the following sections, the first-order and the second-order eccentricity perturbation effects on the orbit are analyzed in turn.

First-Order Averaging

By applying Hamilton's equations and averaging the first-order terms, the following differential equations for the variation of mean action-angle variables are obtained:

$$\begin{aligned}
\dot{\bar{\beta}}_1 &= -\frac{3\alpha_3 \sin(2\beta_1)}{2\sqrt{\alpha_1}}e \\
\dot{\bar{\beta}}_2 &= 0 \\
\dot{\bar{Q}}_3 &= -3\alpha_3 - 3\sqrt{\alpha_1}\sin(2\beta_1)e \\
\dot{\bar{\alpha}}_1 &= 6\alpha_3\sqrt{\alpha_1}\cos(2\beta_1)e \\
\dot{\bar{\alpha}}_2 &= 0 \\
\dot{\bar{\alpha}}_3 &= 0
\end{aligned} \tag{21}$$

In the above equations, α_1 defines the size of the in-plane motion, and β_1 is the associated phase angle. α_2 and β_2 define the out-of-plane motion. α_3 and Q_3 are the drifts in the x and y directions, respectively.

Examining these equations, the following observations can be made:

1. As expected, the first-order eccentricity perturbation does not lead to out-of-plane motion ($\dot{\bar{\alpha}}_2 = 0$, $\dot{\bar{\beta}}_2 = 0$).

2. When $\alpha_3 = 2x_0 + \dot{y}_0 = 0$ and $\sin(2\beta_1) = -3x_0 - 2\dot{y}_0 = 0$, there is no secular, in-plane drift. This corresponds to the satellite being on the y-axis at perigee.

3. With these constraints, the size of the relative motion orbit and the phase shift stay constant ($\dot{\bar{\alpha}}_1 = 0$, $\dot{\bar{\beta}}_1 = 0$).

Second-Order Averaging

In second-order averaging, the effect of the fast terms that were averaged away in the first order analysis must be taken into account. Following the second-order averaging theorem in Sanders[8], the following initial value problem is considered:

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x) + \varepsilon^3 R(t,x,\varepsilon)\ , \quad x(0) = x_0 \tag{22}$$

The corresponding averaged differential equation becomes:

$$\dot{u} = \varepsilon\ f^o(u) + \varepsilon^2 f^{1o}(u) + \varepsilon^2 g^o(u)\ , \quad u(0) = x_0 \tag{23}$$

where f^o, f^{1o} and g^o represent the average values of f, f^1 and g.

Fast variations of the first-order term in ε affect the second-order long-term variations in ε^2 via f^1, which can be computed from the following two equations:

$$\begin{aligned} f^1 &= \nabla f(t,x)u^1(t,x) - \nabla u^1(t,x) f^o(x) \\ u^1 &= \int \left(f(\tau,x) - f^o(x) \right) d\tau + a(x) \end{aligned} \tag{24}$$

with *a(x)* a smooth vector field, this forces u^1 to be zero on average.

Applying the second-order averaging theory to the modified epicyclic elements, we obtain:

$$\begin{aligned} \dot{\bar{\beta}}_1 &= -\frac{3\alpha_3 \sin(2\beta_1)}{2\sqrt{\alpha_1}} e + \left(\frac{-6\alpha_1 - (1-3\alpha_3)^2 + 6\alpha_1 \cos(4\beta_1)}{8\alpha_1} \right) e^2 \\ \dot{\bar{\beta}}_2 &= 0 \\ \dot{\bar{Q}}_3 &= -3\alpha_3 - 3\sqrt{\alpha_1} \sin(2\beta_1) e - \frac{3}{4}(9 + 19\alpha_3) e^2 \\ \dot{\bar{\alpha}}_1 &= 6\alpha_3 \sqrt{\alpha_1} \cos(2\beta_1) e + 6\alpha_1 \sin(2\beta_1) \cos(2\beta_1) e^2 \\ \dot{\bar{\alpha}}_2 &= 0 \\ \dot{\bar{\alpha}}_3 &= -\frac{3Q_3}{2} e^2 \end{aligned} \tag{25}$$

Analyzing the second-order equations, the following remarks can be made:

1. The second-order eccentricity perturbation also does not lead to out-of-plane motion.

2. Setting $\dot{\bar{Q}}_3, \dot{\bar{\alpha}}_3, \dot{\bar{\alpha}}_1$ equal to zero gives us: $Q_3 = -2 \,|\, \dot{x}_0 \,| + y_0 = 0$ and two options for the other two equations.

The first alternative is to set $\alpha_3 = 0$ and $\sin(2\beta_1) = -3x - 2\dot{y} = 0$. This is equivalent to the first-order solution, which means that initially, at the perigee, the follower is on the y-axis and $\dot{x}_0 = y_0 / 2$.

In the second alternative, $\cos(2\beta_1) = \dot{x} = 0$ and $-3\alpha_3 - 3\sqrt{\alpha_1}\sin(2\beta_1)e - \frac{3}{4}(9 + 19\alpha_3)e^2 = 0$. The first equation combined with the second condition means that, at the perigee, the satellite is on the x-axis. The second equation gives an approximation of the general period matching constraint, the semi-major axis being equal[2], which can be approximated by the following equation[9]:

$$\frac{\dot{y}_0}{x_0} = -\frac{n(2+e)}{(1+e)^{1/2}(1-e)^{3/2}} \tag{26}$$

These alternatives yield the two extreme cases in the periodic in-plane solution of the C-W equation ($x = \rho \sin(t + \varphi_o)/2$, $y = \rho \cos(t + \varphi_o)$ where φ_o defines the initial angle between x and y). The first alternative puts the satellite initially on the apoapsis, while the second alternative puts it on the periapsis (corresponding respectively to the grey and black points in Figure 3).

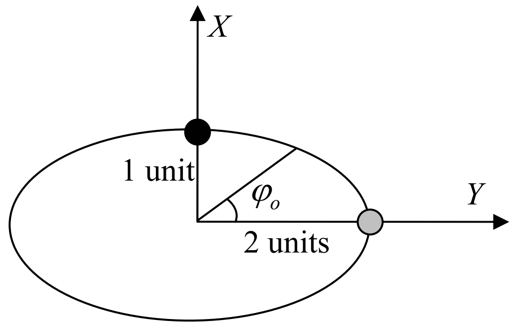

Figure 3 C-W periodic relative in-plane motion ellipse

Both alternatives have their advantages and disadvantages. If the follower is initially located on the apoapsis, then no drift in the y-direction occurs (to second order) and no correction is needed. However, the higher-order terms in the approximations of $\dot{\alpha}_1$ all include $\cos(2\beta_1)$. This demonstrates that the size of the ellipse changes over time (see Figure 4).

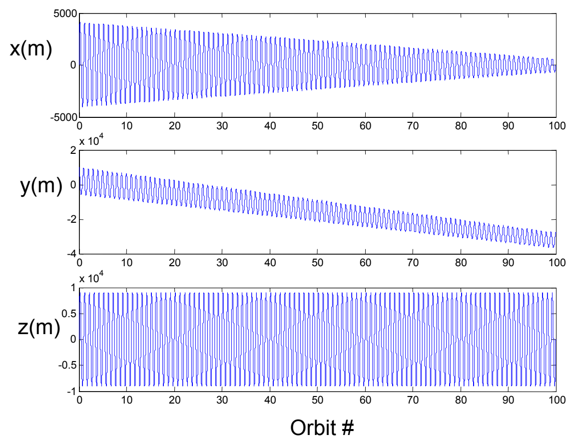

Figure 4 Components of the relative distance for initial condition $\varphi_o = 0$, *e*=0.1.

If the follower is located on the periapsis, a correction to the initial in-track velocity, $\dot{y}_o$, is needed. This sets α_{3o} equal to a non-zero value (see Eq. (25)); leading the satellite to drift slowly with the along-track drift, thus moving the origin of the ellipse in time (see Figure 5).

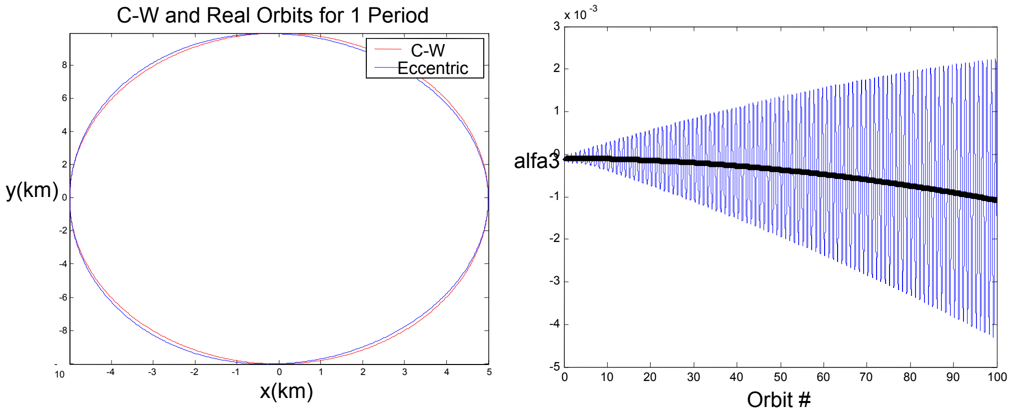

Figure 5 Radial drift in the relative motion ellipse for $\varphi_o = 90$, *e*=0.1 with boundedness correction.

For the initial conditions between the periapsis and the apoapsis, the intensity of these effects is lessened and they occur in combination.

CONCLUSION

A Hamiltonian approach to solve for the relative motion in an eccentric reference orbit has been applied. A new set of action-angle variables that define the physics of the motion have been found. Using Canonical Perturbation Theory, the evolution of the action-angle variables in time was found. Applying the second order averaging theory, the variations were orbit averaged and the periodicity and boundedness conditions were obtained. The comparison of the different initial conditions was done analytically and with simulations.

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